

NEW REAL VARIABLE METHODS IN H SUMMABILITY OF FOURIER SERIES.

CALIXTO P. CALDERÓN, A. SUSANA CORÉ, AND WILFREDO O. URBINA

ABSTRACT. In this paper we shall be concerned with H_α summability, for $0 < \alpha \leq 2$ of the Fourier series of arbitrary $L^1([-\pi, \pi])$ functions. The method to be employed is a refinement of the real variable method introduced by Marcinkiewicz in [8].

Dedicated to the memory of A. Eduardo Gatto

1. INTRODUCTION

Let f be a function in $L^1([-\pi, \pi])$, denote by $S_n(f, \cdot)$ the partial sum of order n of the Fourier series of f ,

$$(1.1) \quad S_n(f, x) = \sum_{|k| \leq n} c_k e^{-ikx}, \quad x \in [-\pi, \pi].$$

We say that f is H_2 summable at x if there exists a number s such that,

$$\frac{1}{n} \sum_{k=1}^n |S_k(f, x) - f(x)|^2 \longrightarrow 0 \quad \text{a.e.}$$

This can be extended easily to $\alpha > 0$; i.e. we say that its Fourier series is H_α summable to some $f(x)$ or that it is a strongly α -summable to sum $f(x)$, if

$$(1.2) \quad \frac{1}{n} \sum_{k=1}^n |S_k(f, x) - f(x)|^\alpha \longrightarrow 0 \quad \text{a.e.}$$

Historically the problem goes back to H. Hardy and J. H. Littlewood in [6]. There the problem is restricted to H_2 summability of $L^2([-\pi, \pi])$ functions (i.e. $\alpha = 2$ and $f \in L^2([-\pi, \pi])$), see also T. Carleman [5].

In 1935 Hardy and Littlewood proved the case $\alpha > 0$ and $f \in L^p$, with $1 < p < \infty$ and posed the problem of whether “any arbitrary periodic function in $L^1([-\pi, \pi])$ is H_2 summable a. e. in $[-\pi, \pi]$ ”. The answer to this question came only on January of 1939, when J. Marcinkiewicz presented his remarkable result [8], developing a real variable method to establish it.

Finally the case of H_α summability a. e. for $\alpha > 2$ and $f \in L^1([-\pi, \pi])$ was proved by A. Zygmund in 1941, [12] using complex methods. In view of the negative results concerning convergence a.e. of the Fourier series of functions in $L^1([-\pi, \pi])$ the H_α summability

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acquires a special meaning.

In this paper we shall be concerned with H_α summability, for $0 < \alpha \leq 2$ of Fourier series for arbitrary $L^1([-\pi, \pi])$ functions. The methods used here are a refinement of the real variable method by Marcinkiewicz in [8], and could be applied also to the case $\alpha > 2$. Nevertheless this requires a modification of the Marcinkiewicz function and a change of kernel function (to be defined later).

2. PRELIMINARIES

Consider the following maximal operator

$$(2.1) \quad (\sigma_\alpha^* f)(x) = \sup_{n>0} \left[\frac{|S_1(f, x)|^\alpha + |S_2(f, x)|^\alpha + \dots + |S_n(f, x)|^\alpha}{n} \right]^{1/\alpha} \\ = \sup_n \sqrt[n]{\frac{1}{n} \sum_{k=1}^n |S_k(f, x)|^\alpha},$$

where, as before, $S_k(f, \cdot)$ stands for the k -th partial sum of the Fourier series of $f \in L^1([-\pi, \pi])$ and $0 < \alpha \leq 2$.

Also, let us consider for $f \in L^1([-\pi, \pi])$ the non-centered Hardy-Littlewood function, namely,

$$(2.2) \quad f^*(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| dy$$

where I is taken to be an open interval, containing x . Observe that the set

$$F = \{x : f^*(x) \leq \lambda\}$$

is a closed set, and the set

$$G = \{x : f^*(x) > \lambda\},$$

is an open set.

The class A_1 of weights is defined using the non-centered Hardy-Littlewood function, f^* , we say $\omega \in A_1$ if the inequality

$$(2.3) \quad \omega(\{x : f^*(x) > \lambda\}) \leq \frac{C}{\lambda} \int_{-\pi}^{\pi} |f(x)| \omega(x) dx,$$

holds true for any $f \in L^1([-\pi, \pi])$, $f \geq 0$ C depends only on ω .

A well known result gives a characterization of the weights in the case $(-\infty, \infty)$, see Stein [11]; a positive weight $w \geq 0$ belongs to the class A_1 if and only if

$$(2.4) \quad \omega^*(x) \leq C\omega(x).$$

In order to prove the problem of H_2 summability for L^1 functions Marcinkiewicz proved that σ_2^* is finite a.e. and he refined that to H_2 summability. Moreover, it can be proved, see Stein [9],

$$(2.5) \quad |\{x : (\sigma_2^* f)(x) > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1,$$

C depends only on ω .

In order to tackle that problem Marcinkiewicz introduced the so called *Marcinkiewicz function*. If F is a perfect set and $G = F^c$ its complement, if $d(x, F) = \inf_{z \in F} |x - z|$, denotes the distance from x to F ; then he defined,

$$(2.6) \quad \mathcal{F}(x) = \int_G \frac{1}{(x - y)^2} d(x, F) dy$$

which is finite a.e. for $x \in F$.

The function $\mathcal{F}(x)$ has important implications in the L^1 theory of singular integrals. In particular, in 1966, L. Carleson in his famous L^2 theorem uses a variation of this function, Carleson function is denoted as Δ in his article. Also Zygmund shows the relation between $\Delta(x)$ and $\mathcal{F}(x)$, see [13].

By Kolmogorov's counterexample in 1926, we know that there exists a function $f \in L^1([-\pi, \pi])$ such that $S_n(f, \cdot)$ diverges a.e., thus the maximal function,

$$f^{**}(x) = \sup_n |S_n(f, x)|$$

can not be weak type $(1, 1)$, see Zygmund [14], Vol II.

Now, consider firstly functions supported on an interval of length $\pi/8$, centered at the origin. For those functions the maximal function $\sigma_\alpha^* f$ satisfies the inequality

$$(\sigma_\alpha^* f)(x) \leq C f^*(x),$$

for x such that $d(0, x) \geq \pi/4$, where C is a constant depending on π only. The above inequality is consequence of

$$\int_{-\pi}^{\pi} |f(t)| dt \leq 2\pi f^*(x),$$

and the estimate

$$\left| \frac{\sin(n + 1/2)u}{\sin u/2} \right| \leq \frac{1}{|u|} + \frac{1}{2},$$

see A. Zygmund [14], Vol I pages 50–51. In what follows we shall introduce a majorization of the kernel used by Marcinkiewicz in [8].

Returning to the case of a general function f , such a function can be decomposed into a sum of pieces, each supported on an interval of length $\pi/8$. By a shift each one of the pieces is moved to the origin, thus each piece can be studied as if it were supported on an interval of length $\pi/8$ centered at the origin.

The method to be employed in this paper is a refinement of the real variable method introduced by Marcinkiewicz in [8].

3. MAIN RESULTS

The main results obtained in this paper are the following,

Theorem 3.1. *Given $0 < \alpha \leq 2$, $f \in L^1([-\pi, \pi])$ and $\sigma_\alpha^* f$ as above, then*

$$(3.1) \quad |\{x : (\sigma_\alpha^* f)(x) > \lambda\}| \leq \frac{C_\alpha}{\lambda} \|f\|_1,$$

where C_α is a constant that depends only on the α (but not on f).

Moreover, if ω is an A_1 -weight, then

$$(3.2) \quad \omega\left(|\{x : (\sigma_\alpha^* f)(x) > \lambda\}|\right) \leq \frac{C_\omega}{\lambda} \int_{-\pi}^{\pi} |f(x)| \omega(x) dx,$$

C_ω is a constant that depends only on the weight (but not on f).

As a particular case, if $\omega(x) \equiv 1$ we have the Lebesgue's measure case.

The following result was given by J. Marcinkiewicz in [8] (see Lemma 3),

Lemma 3.1. *Let P be a perfect set (i.e., a set without isolated points), Δ_v a sequence of contiguous segments, $\varphi(x)$ the function of period 2π equal to zero in P and to $|\Delta_v|$ for $x \in \Delta_v$. We have almost everywhere in P*

$$(3.3) \quad \int_{-\pi}^{\pi} \frac{\varphi(t+x)}{t^2} dt < \infty.$$

For the proof see [8]. We will give an alternative proof of this result.

3.1. Whitney type decomposition. We start considering an special type of covering for open sets in \mathbb{R} which has the same type of building principle than the Whitney decomposition in \mathbb{R}^n , see Stein [10],

Lemma 3.2. *Let G be an open set, and consider its decomposition into open disjoint intervals $\{J_k\}$ (i.e. G can be written as $G = \bigcup_k J_k$, and $J_k \cap J_l = \emptyset$ for $k \neq l$) such that J_k are its connected components and the end points of J_k are in $F = G^c$. Then it is possible to find a countable refinement $\{I_j\}$ such that,*

i) *whenever I_j and $I_{j'}$ are not adjacent, i.e. $\bar{I}_j \cap \bar{I}_{j'} = \emptyset$ then, for a suitable C*

$$d(I_j, I_{j'}) \geq C|I_j| \text{ and}$$

$$d(I_j, I_{j'}) \geq C|I_{j'}|.$$

C can be chosen to be bigger or equal to $1/2$.

ii) *For any j , $d(I_j, F) = |I_j|$.*

Proof. We have $G = \bigcup_k J_k = \bigcup_k (a_k, b_k)$ with $a_k, b_k \in F = G^c$. For each fixed k we do the following:

- We decompose J_k into three subintervals $J_{k,1}$, $J_{k,2}$, $J_{k,3}$ having equal length, thus

$$J_k = \bigcup_{i=1}^3 J_{k,i} \quad \text{and} \quad d(J_{k,i}, F) = |J_{k,i}|$$

$J_{k,2}$ is selected to be a closed interval and $J_{k,1}, J_{k,3}$ are open intervals.

- $J_{k,2}$, the central subinterval, will be an element of the new refinement $\{I_j\}$.
- Each of the side open intervals, $J_{k,1}$ and $J_{k,3}$ are broken up into two subintervals of the same length such that, the ones that are adjacent to the central interval $J_{k,2}$, are taken such that the left one $J_{k,1,2}$ is closed on the left and open on the right and the right one $J_{k,3,1}$ is open on the left and closed to the right and therefore

$$\begin{aligned} d(J_{k,1,2}, F) &= |J_{k,1,i}| & i &= 1, 2 \\ d(J_{k,3,1}, F) &= |J_{k,3,i}| & i &= 1, 2 \end{aligned}$$



The intervals, $J_{k,1,2}$ and $J_{k,3,1}$ will be part of the new refinement $\{I_j\}$.

- As before, the remaining side open intervals $J_{k,1,1}$ and $J_{k,3,2}$ are broken up into two subintervals of the same length, the one that is adjacent to the interval $J_{k,1,2}$, is taken such that is closed on the left and open on the right and the one that is adjacent to the interval $J_{k,3,1}$ is open on the left and closed to the right and they will be part of the new refinement $\{I_j\}$.
- Iterating this argument over and over again and doing the same process for each J_k of the original decomposition of G we obtain a sequence of intervals $\{I_j\}$, such that $d(I_j, F) = |I_j|$.

It is important to note that if I_j and $I_{j'}$ are not adjacent, i.e. $\bar{I}_j \cap \bar{I}_{j'} = \emptyset$, then there will be among them at least one subinterval satisfying the construction conditions, and therefore they satisfy

$$\begin{aligned} d(I_j, I_{j'}) &\geq C|I_j| \\ &\geq C|I_{j'}| \end{aligned}$$

□

3.2. Consequences of this Whitney type decomposition. Now, given $f \in L^1[-\pi, \pi]$, $f \geq 0$ and $\lambda > 0$ consider the set

$$G = \{x : f^*(x) = \sup \frac{1}{|I|} \int_I f(t) dt > \lambda\},$$

then G is an open set, and consider the Whitney type decomposition for G , $\{I_j\}$, as above, i.e. $G = \cup_{j=1}^{\infty} I_j$. We take its average

$$\frac{1}{|I_j|} \int_{I_j} f dx \leq \frac{1}{|\tilde{I}_j|} \int_{\tilde{I}_j} f dt$$

where \tilde{I}_j has been obtained from I_j by expanding it 2 times, i.e. $|\tilde{I}_j| \geq 2|I_j|$. If I_j is one of the central subintervals of the original decomposition $J_{k,2}$, one of the adjacent subintervals is

also included. If I_j is not the central subintervals, we choose another subinterval, adjacent to the central one. In this way we have $\tilde{I}_j = I_j \cup I$ and therefore $|\tilde{I}_j| = 2|I_j|$

$$\begin{aligned} \frac{1}{|I_j|} \int_{I_j} f dx &\leq \frac{1}{|I_j|} \int_{\tilde{I}_j} f dt = \frac{2}{2|I_j|} \int_{\tilde{I}_j} f dt \\ &\leq \frac{1}{|\tilde{I}_j|} \int_{\tilde{I}_j} f dt \leq 2\lambda. \end{aligned}$$

Regardless of I_j , the \tilde{I}_j we have defined has points from the complement of G , and therefore its integral is less or equal than 2λ . In other words,

$$\int_{I_j} f dx \leq 2\lambda|I_j| \quad \text{if and only if} \quad \int_{\tilde{I}_j} f dx \leq 2\lambda|\tilde{I}_j|$$

given that \tilde{I}_j contains at least one point from $F = G^c = \{f^* \leq \lambda\}$.

Suppose now that we have a Poisson kernel and a function f such that $f \geq 0$, $\text{supp}(f) \subset J_k$ and f is *bad*, by which we mean that f is infinite in a dense subset (i.e., for all n there exists $E_n \subset J_k$ such that $|f| > n$). Even though f is bad, we know that for some k_0

$$\frac{1}{|J_{k_0}|} \int_{J_{k_0}} f dx \leq \lambda \quad \text{and} \quad d(u, J_{k_0}) \geq c|J_{k_0}|$$

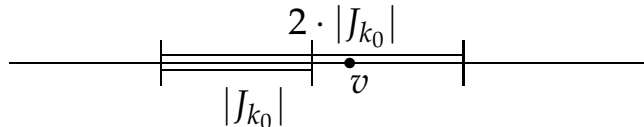
$$(3.4) \quad \int_{J_{k_0}} \frac{\varepsilon}{\varepsilon^2 + (u-v)^2} f(v) dv \leq C(c)\lambda$$

$$(3.5) \quad \int_{J_{k_0}} \frac{\varepsilon}{\varepsilon^2 + (u-v)^2} f(v) dv \leq C \int_{J_{k_0}} \frac{\varepsilon}{\varepsilon^2 + (u-v)^2} \phi(v) dv$$

where $\phi(v) \leq C(c)\lambda$ and $|u-v| > c|J_{k_0}|$.

Proof. We will prove (3.2). Take $v \in J_{k_0}$ such that $d(u, J_{k_0}) > C|J_{k_0}|$,

$$\begin{aligned} \int_{J_{k_0}} \frac{\varepsilon}{\varepsilon^2 + (u-v)^2} f(v) dv &\leq \int_{J_{k_0}} \frac{\varepsilon}{\varepsilon^2 + c^2|J_{k_0}|^2} f(v) dv \quad \text{if } |u-v| < C|J_{k_0}| \\ &\leq \frac{\varepsilon}{\varepsilon^2 + c^2|J_{k_0}|^2} \int_{J_{k_0}} f(v) dv = \frac{\varepsilon|J_{k_0}|}{\varepsilon^2 + c^2|J_{k_0}|^2} \frac{1}{|J_{k_0}|} \int_{J_{k_0}} f(v) dv \\ &\leq \frac{\varepsilon|J_{k_0}|}{\varepsilon^2 + c^2|J_{k_0}|^2} \lambda = \frac{\varepsilon\lambda}{\varepsilon^2 + \frac{c^2}{4}(2|J_{k_0}|^2)^2} |J_{k_0}| \\ &\leq \frac{\varepsilon\lambda}{\varepsilon^2 + \frac{c^2}{4}(|J_{k_0}|^2 + u)^2} \end{aligned}$$



Therefore this must hold for the average on v

$$\begin{aligned} \int_{J_{k_0}} \frac{\varepsilon}{\varepsilon^2 + (u - v)^2} f(v) dv &\leq \varepsilon \lambda |J_{k_0}| \frac{1}{|J_{k_0}|} \int_{J_{k_0}} \frac{dv}{\varepsilon^2 + \frac{c^2}{4} (|I_{k_0}| + v)^2} \\ &= \int_{J_{k_0}} \frac{\varepsilon}{\varepsilon^2 + \frac{c^2}{4} (|I_{k_0}| + v)^2} \lambda dv = C \lambda \end{aligned}$$

□

(3.4) was the key point in Marcinkiewicz's proof.

3.3. On the Marcinkiewicz function. Let $\mathcal{F}(x)$ be the Marcinkiewicz function, defined in (2.6) and $\{I_k\}$ the covering of G satisfying the properties of Lemma 3.2 and consider

$$(3.6) \quad \sum_{k=1}^{\infty} \left(\int_{I_k} \frac{|I_k|}{(x - y)^2} f_k dy \right)$$

where $f_k = \frac{1}{|I_k|} \int_{I_k} f(t) dt$. Then it is not difficult to see that if $x \in F = G^c$, then (3.6) is finite and

$$\begin{aligned} \int_F \left[\sum_{k=1}^{\infty} \left(\int_{I_k} \frac{|I_k|}{(x - y)^2} f_k dy \right) \right] &= \sum_{k=1}^{\infty} \int_{I_k} \left[f_k (|I_k| \int \frac{1}{(x - y)^2} dx) \right] dy \\ &\leq C \sum_{k=1}^{\infty} \int_{I_k} f_k dy \\ &\leq \lambda C |G| \end{aligned}$$

where the last inequality follows from the construction of the covering and the first reflects the fact that $|I_k| \int \frac{1}{(x - y)^2} dx \leq C$ because $|I_k| \int \frac{1}{x^2} dx \leq C$ for $|x| \geq |I_k|$. Furthermore

$$\begin{aligned} \left| \{ \mathcal{F}(x) > \lambda \} \right| &\leq \frac{C}{\lambda} \sum_k \left(\int f_k(y) dy \right) \\ &\leq \frac{C}{\lambda} \lambda |G| \\ &\leq C \frac{1}{\lambda} \int f(x) dx \end{aligned}$$

In other words, the Marcinkiewicz function \mathcal{F} is $(1, 1)$ -weak.

Proposition 3.1. Let μ be a measure such that $\mu \in A_1$ and $d\mu = g dx$, where g is the density of the measure μ . Then

$$\int_G g dx \leq \frac{C}{\lambda} \int f g dx$$

Proof. Let $\mathcal{F}(x)$ be the Marcinkiewicz function

$$\begin{aligned} \int \mathcal{F}(x) d\mu(x) &\leq \int_F d\mu(x) \int \frac{1}{(x-y)^2} |I_k| f_k(y) dy \\ &\leq \int_{I_k} f(y) \left[\frac{d\mu(x)}{[(y-x) + |I_k|]^2} |I_k| \right] dy \\ &\leq \mathcal{F}g(y) \leq C g(y) \leq C \int f(y)g(y) dy \end{aligned}$$

since $(x-y)^2 \sim [(y-x) + |I_k|]^2$. Therefore using Chebyshev's inequality

$$\left(\mu\{x : \mathcal{F}(x) > \lambda\} \right) \leq \frac{C}{\lambda} \int_F \mathcal{F}(x) d\mu(x)$$

over F . Then by Marcinkiewicz's theorem

$$\mu(G) \leq \frac{C}{\lambda} \int f(y)g(y) dy.$$

□

Theorem 3.2 (surprising result).

$$(3.7) \quad \int \frac{|f(x+y)|}{y^2} dy < \infty \quad a.e. \text{ in } F$$

Proof. Let $f_G = f|_G$ and $G = \bigcup_k J_k$, where J_k denotes maximal intervals, and $\tilde{J}_k \cap F \neq \emptyset$. We shall see that

$$(3.8) \quad \int \frac{1}{(x-y)^2} f_G(y) dy = \int \frac{f_G(x+y)}{y^2} dy < \infty \quad a.e.$$

We shall show next that

$$\sum_k \int_{J_k} f_G(y) dy \int_F \frac{1}{(x-y)^2} dx$$

Consider $\tilde{J}_k = (1+\varepsilon)J_k$, a dilation by a factor of $1+\varepsilon$ from the center of J_k , therefore

$$|\tilde{J}_k| = |(1+\varepsilon)J_k| = (1+\varepsilon)|J_k|$$

If $x \in \tilde{F} = \left(\bigcup (1+\varepsilon)J_k \right)^c \subset F$ then

$$\int_{\tilde{F}} \frac{1}{(x-y)^2} dx \leq \int_{|x-y| > \varepsilon |J_k|} \frac{1}{(x-y)^2} dy \leq \frac{1}{\varepsilon |J_k|}$$

Then

$$\int_{J_k} f_G(y) \left(\int_{\tilde{F}} \frac{dx}{(x-y)^2} \right) dy \leq \frac{1}{\varepsilon |J_k|} \int_{J_k} f_G(y) dy \leq \frac{1}{\varepsilon} \lambda$$

We observe that when x is outside the intervals I_k , that is whenever $x \in \tilde{F}$, the integral $\int_{I_k} \frac{1}{(x-y)^2} f_G(y) dy$ is bounded from below and from above. If $x \notin I_k$ $y_k \in I_k$ and if we note that $d(x, I_k) \geq \varepsilon |I_k|$

$$\begin{aligned} \int_{I_k} \frac{1}{(x-y)^2} f_G(y) dy &\sim \frac{1}{(x-y_k)^2} \int_{I_k} f_G(y) dy \\ &= \frac{|I_k|}{(x-y_k)^2} \frac{1}{|I_k|} \int_{I_k} f_G(y) dy \\ &\leq \underbrace{\frac{1}{|I_k|} \int_{I_k} f_G(y) dy}_{=\lambda} \int_{I_k} \frac{1}{(x-y)^2} dy \\ &= \lambda \int_{I_k} \frac{1}{(x-y)^2} dy \end{aligned}$$

Summing up over k we obtain

$$\sum_k \int_{I_k} \left(\int_{\tilde{F}} \frac{dx}{(x-y)^2} \right) dy \leq C\lambda \int_{\tilde{G}} \frac{1}{(x-y)^2} dy.$$

□

Now we shift our attention on a more pure version. If $G = \bigcup I_k$, $F = G^c$, $\delta_k = c|I_k|$ and defining ϕ by

$$(3.9) \quad \phi(x) = \begin{cases} c|I_k| & \text{if } x \in I_k \\ 0 & \text{if } x \in F \end{cases}$$

we have

$$(3.10) \quad \phi(x) = \sum_k c|I_k| \chi_{I_k}(x)$$

Lemma 3.3.

$$(3.11) \quad \int_{F'} \frac{1}{(x-y)^2} \phi(y) dy < \infty \quad \text{a.e. on } F$$

Proof. Let $F' = (G')^c$, where $G' = \bigcup_k (1+\varepsilon)I_k$. Denoting by $I'_k = (1+\varepsilon)I_k$

$$\begin{aligned} \int_{F'} \int \frac{1}{(x-y)^2} \phi(y) dy dx &= \sum_k \int_{I'_k} \sum_k c_k |I_k| \int_{F'} \frac{1}{(x-y)^2} dx dy \\ (3.12) \quad &= \sum_k \int_{I_k} c dy \\ &= C'' |G| \end{aligned}$$

where we have used that $(x - y)^2 \geq (1 + \varepsilon)|I_k|$ if $x \in F'$. Therefore

$$(3.13) \quad \begin{aligned} \int_{F'} \left(\int \frac{1}{(x - y)^2} \phi(y) dy \right) dx &< \infty \\ \int \frac{1}{(x - y)^2} \phi(y) dy &< \infty \quad \text{a.e. on } F \end{aligned}$$

□

We plan to discuss the following chain of inequalities

$$(3.14) \quad \lambda \leq \frac{1}{|I_k|} \int_{I_k} f(u) du \leq 2\lambda$$

For $f \geq 0$, $u \geq 0$ and $v \geq 0$

$$\begin{aligned} (1 - r^2) \int_0^{\frac{\pi}{2}} \frac{f(u + x)}{(1 - r)^2(u + v)^2} \int_0^{\pi/2} \frac{f(v + x)}{(1 - r)^2(u - v)^2} dv du \\ \leq (1 - r^2) \int_0^{\frac{\pi}{2}} \frac{f(u + x)}{(1 - r)^2 u^2} \int_0^{\frac{\pi}{2}} \frac{f(v + x)}{(1 - r)^2(u - v)^2} dv du \quad \text{since } v \geq 0 \\ \leq (1 - r^2) \int_x^{x + \frac{\pi}{2}} \frac{f(\bar{u})}{(1 - r)^2(\bar{u} - x)^2} \int_0^{\frac{\pi}{2}} \frac{f(\bar{v})}{(1 - r)^2(\bar{u} - \bar{v})^2} d\bar{v} d\bar{u} \\ \leq (1 - r^2) \int_{-\pi}^{-\pi} \frac{f(u)}{(1 - r)^2(u - x)^2} \int_{-\pi}^{\pi} \frac{f(v)}{(1 - r)^2(u - v)^2} dv du \end{aligned}$$

Then

$$\begin{aligned} (1 - r^2) \int_{-\pi}^{\pi} \frac{f_k(u)}{(1 - r)(u - x)^2} \sum_{\substack{j=k' \\ xx=k''}} \int_{-\pi}^{\pi} \frac{f_j(\cdot)}{(1 - r)(u - v)^2} du dv \\ \leq (1 - r^2) \int_{-\pi}^{\pi} \frac{f_k(u)}{(1 - r)^2(u - x)^2} \sum_{\substack{j=k' \\ xx=k''}} \frac{f_j(\cdot)}{(1 - r)^2} dv du \\ \leq \int_{-\pi}^{\pi} \frac{f_k(u)}{(u - x)^2} C\lambda |I_k| du \end{aligned}$$

Let

$$f_k = \begin{cases} f & \text{on } I_k \\ 0 & \text{elsewhere.} \end{cases}$$

If $x \in I_j$, then

$$\begin{aligned}
 \varphi_k(x) &= \int_{\chi_k} \frac{1-r}{(x-y)^2 + (1-r)^2} f_k(y) dy \\
 &\leq \int_{\chi_k} \frac{1+r}{(1-r)^2 + c(x-y)^2} \mu_k dy \\
 \int \frac{1-r}{(1-r)^2 + (x-y)^2} f_k(y) dy &\leq \left(\frac{1-r}{(1-r)^2 + (x-y)^2} dy \right) \mu_k \\
 &\leq \left(\frac{1-r}{(1-r)^2 + C(x-y)^2} dy \right) \mu_k \\
 \int \frac{\varepsilon^{p-1}}{\varepsilon^p + (x-y)^p} dy &= \int \frac{\varepsilon^{p-1}}{\varepsilon^p + |y|^p} dy = \int \frac{\varepsilon^{p-1}}{1 + \left| \frac{y}{\varepsilon} \right|^p} dy \\
 &= \int \frac{\varepsilon^{-1}}{1 + \left| \frac{y}{\varepsilon} \right|^p} dy = \int \frac{ds}{1 + |s|^p}, \quad \frac{y}{\varepsilon} = s.
 \end{aligned}$$

3.4. Power Series. Now, consider the power series, $\sum_{k=0}^{\infty} a_k x^k$, for x a complex variable, which is convergent for $|x| < 1$ and let $f(x) = \sum_{k=0}^{\infty} a_k x^k$. Then f is analytic.

Denoting the partial sum of $\sum_{k=0}^{\infty} a_k$ by $S_n = \sum_{k=0}^n a_k$, then, as $S_k - S_{k-1} = a_k$, then

$$\begin{aligned}
 (3.15) \quad f(x) &= \sum_{k=0}^{\infty} (S_k - S_{k-1}) x^k = \sum_{k=0}^{\infty} S_k x^k - \sum_{k=1}^{\infty} S_{k-1} x^k \\
 &= \sum_{k=0}^{\infty} S_k x^k - \sum_{k=0}^{\infty} S_k x^{k+1} = \sum_{k=0}^{\infty} S_k (x^{k+1} - x^k) = (1-x) \sum_{k=0}^{\infty} S_k x^k
 \end{aligned}$$

Suppose now $x \in B_1 = \{x : |x| < 1\}$, then

$$(3.16) \quad \frac{f(x)}{1-x} = \sum_{k=0}^{\infty} S_k x^k$$

Writing $x^k = r^k e^{ik\theta}$,

$$(3.17) \quad \frac{f(x)}{1-x} = \sum_{k=0}^{\infty} (S_k r^k) e^{ik\theta}$$

If $1 < p < 2$, then using the Hausdorff-Young inequality we have

$$\begin{aligned}
 (3.18) \quad \left[\sum_{k=0}^{\infty} |S_k r^k|^q \right]^{1/q} &\leq C_p \left(\int_0^{2\pi} \left| \frac{f(re^{i\theta})}{1-re^{i\theta}} \right| d\theta \right)^{1/p} \\
 &\quad \times \left[\int_0^{2\pi} \frac{1}{1-re^{i\theta}} \left(\int_0^{2\pi} |P(r, \theta - \psi)| f(\psi) d\psi \right)^p \right]^{1/p}
 \end{aligned}$$

Now,

$$\sum_{k=0}^{\infty} a_k e^{iks} e^{ik\theta} r^k = f(re^{i(\theta+s)})$$

$$f(r, \theta + s) = \sum_{k=0}^{\infty} (1-z) S_k z^k \quad \text{where } z = e^{is} r \text{ and } S_n = \sum_{k=0}^n a_k e^{ik\theta}.$$

Now θ has been fixed and s varies, therefore

$$\sum_{k=0}^{\infty} S_k z^k = \frac{1}{1-z} f(r, \theta + s).$$

By the Hausdorff-Young inequality which bounds the L^p norm the Fourier coefficients for $1 < p < 2$ and $2 < q < \infty$

$$(3.19) \quad \left(\sum_{k=0}^{\infty} r^k q |S_k|^q \right)^{1/q} \leq C_p \left(\int_0^{2\pi} \frac{1}{|1 - re^{is}|^p} f(r, \theta + s)^p ds \right)^{1/p}.$$

Now, since we have the equivalence $|1 - re^{is}| \sim |(1-r)^2 + s^2|^{1/2}$ in the sense it is bounded by (3.19) multiplied by a constant

$$\begin{aligned} &\leq \left(\int_0^{2\pi} \frac{1}{[(1-r)^2 + s^2]^{p/2}} \left(\int P(r, \theta + s - u) p(u) du \right)^p \right)^{1/p} \\ &= \left(\int_0^{2\pi} \frac{1}{[(1-r)^2 + (s - \theta)^2]^{p/2}} \left(\int P(r, \theta + s - u) p(u) du \right)^p \right)^{1/p} \end{aligned}$$

Using a Whitney type decomposition with $f = \sum_{k=0}^{\infty} f_k$ we can rewrite the previous equation as

$$\left(\int_0^{2\pi} \frac{1}{[(1-r)^2 + (s - \theta)^2]^{p/2}} \left(P(r, s - u) f_k(u) \sum_j \left(\int P(r, s - v) f_j(v) dv \right) \right)^p \right)^{p-1}$$

where $1 - r = \varepsilon$ and where for now we shall work with a single k to bound f_k and then we shall consider all k

$$\left(\int_0^{2\pi} \frac{1}{[\varepsilon^2 + (s - \theta)^2]^{p/2}} \left(P(r, s - u) f_k(u) \sum_j \left(\int P(r, s - v) f_j(v) dv \right) \right)^p \right)^{p-1}$$

We consider the different possible cases separately:

Case a: We start with case when the j are not adjacent to I_k .

$$(3.4) \leq \left(\int_0^{2\pi} \frac{1}{[\varepsilon^2 + (s - \theta)^2]^{p/2}} \left(\int P(r, s - v) \lambda \chi_k C \lambda^{p-1} \right) \right)$$

considering s not in I_k

$$\leq \left(\int_0^{2\pi} \frac{1}{[\varepsilon^2 + (s - \theta)^2]^{p/2}} \left(\int P(r, s - v) C \lambda \lambda^{p-1} \right) \right)$$

Case b: In the case that j touch the adjacent, the measures are comparable and we have

$$(3.4) \leq \left(\int_0^{2\pi} \frac{1}{[\varepsilon^2 + (s - \theta)^2]^{p/2}} \left(\int P(r, s - u) C\lambda(\lambda|I_k|)^{p-1} \right) \right)$$

In the case that s not in I_k

$$\begin{aligned} &\leq \left(\int_0^{2\pi} \frac{1}{[\varepsilon^2 + (s - \theta)^2]^{p/2}} \left(\int_{I_k} P(r, s - u) \lambda \chi_k(u) C(\lambda|I_k|)^{p-1} \right) \right) \\ &\leq \left(\int_0^{2\pi} \frac{1}{[\varepsilon^2 + (s - \theta)^2]^{p/2}} \underbrace{\left(\int_{I_k} P(r, s - u) \lambda^p \chi_k(u) C \right)}_{C\lambda^p} \right) \end{aligned}$$

3.4.1. *Abel Sums analog.* Consider the series $(1 - r) \sum_{\nu=0}^{\infty} r^{\nu} (S_{\nu}(f, x))^2$, then taking $r = 1 - \frac{1}{n}$ then

$$\begin{aligned} (1 - r) \sum_{\nu=0}^{\infty} r^{\nu} (S_{\nu}(f, x))^2 &= \left(1 - \left(1 - \frac{1}{n}\right)\right) \sum_{\nu=0}^{\infty} \left(1 - \frac{1}{n}\right)^{\nu} (S_{\nu}(f, x))^2 \\ &= \frac{1}{n} \sum_{\nu=0}^{\infty} \left(1 - \frac{1}{n}\right)^{\nu} (S_{\nu}(f, x))^2 \\ &\geq e^{-1} \frac{1}{n} \sum_{\nu=0}^{\infty} (S_{\nu}(f, x))^2. \end{aligned}$$

Therefore,

$$(S^* f)(x) \leq e^{1/2} \sup_{0 < r < 1} [(1 - r) \sum_{\nu=0}^{\infty} r^{\nu} (S_{\nu}(f, x))^2]^{1/2}.$$

The key will be to study $(1 - r) \sum_{\nu=0}^{\infty} r^{\nu} (S_{\nu}(f, x))^2$. We will construct a kernel

$$(3.20) \quad D(r, x, y) = \sum_{\nu=0}^{\infty} r^{\nu} D_{\nu}(x) D_{\nu}(y),$$

where D_{ν} is the Dirichlet kernel

$$D_{\nu}(x) = \frac{\sin(\nu + 1/2)x}{2 \sin(x/2)}.$$

Thus,

$$D(r, x, y) = \frac{(1 - r)[(1 - r)^2 + 2r(2 + \cos x + \cos y)]}{4(1 - 2r \cos(x - y) + r^2)(1 - 2r \cos(x + y) + r^2)}.$$

For $-\pi + \varepsilon < (x + y) < \pi - \varepsilon$ and $-\pi + \varepsilon < (x - y) < \pi - \varepsilon$ we get,

$$D(r, x, y) \leq \frac{9(1 - r)}{[(1 - r)^2 + rC_3(x - y)^2][(1 - r)^2 + rC_3(x + y)^2]}.$$

We will assume $f = 0$ if $|x| > \pi/2 - \varepsilon/2$. and we will estimate

$$(1 - r) \sum_{\nu=0}^{\infty} r^{\nu} (S_{\nu}(f, x))^2 = \frac{1 - r}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + u) g(x + v) D(x, r, u, v) du dv,$$

We move from the periodic case to the continuous case. The above integral is dominated by

$$(1-r)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{9f(x+u)g(x+v)}{[(1-r)^2 + rc(u-v)^2][(1-r)^2 + rc(u+v)^2]} dudv.$$

We shall consider the integral

$$(3.21) \quad (1-r)^2 \int_0^{\infty} \int_0^{\infty} \frac{f(x+u)g(x+v)}{[(1-r)^2 + rc(u-v)^2][(1-r)^2 + rc(u+v)^2]} dudv,$$

and the analogous integrals on the regions $(-\infty, 0) \times (-\infty, 0)$, $(-\infty, 0) \times (0, \infty)$ and $(0, \infty) \times (-\infty, 0)$, that can be argue as the preceding one using symmetry arguments.

Let us study the integral (3.21), which is typical

$$\begin{aligned} & (1-r)^2 \int_0^{\infty} \int_0^{\infty} \frac{f(x+u)f(x+v)}{[(1-r)^2 + rc(u-v)^2][(1-r)^2 + rc(u+v)^2]} dudv \\ & \leq (1-r)^2 \int_0^{\infty} \int_0^{\infty} \frac{f(x+u)f(x+v)}{[(1-r)^2 + rc(u-v)^2][(1-r)^2 + rcv^2]} dudv \end{aligned}$$

f has already been decomposed as $f = \sum_{k=0}^{\infty} f_k$ where

$$f_k = \begin{cases} f & \text{on } I_k \\ 0 & \text{otherwise.} \end{cases}.$$

$\{I_k\}$ are such that

$$\frac{1}{|I_k|} \int_{I_k} f(t) dt \leq C\lambda.$$

The above integral (3.21) is dominated by

$$C(1-r)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x+u)}{[(1-r) + rc(u-v)^2]} \frac{f(x+v)}{[rv^2]} dudv,$$

and after a change of variables, we have,

$$\frac{c}{r} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(v)}{(v-x)^2} \frac{(1-r)^2 f(u)}{(1-r)} dudv.$$

The intervals I_k have been constructed so that

i) If I_j and I_k are adjacent then

$$\frac{1}{2}|I_j| \leq |I_k| \leq 2|I_j|.$$

ii) $d(I_j, I_k) \geq \frac{1}{2}|I_j|$, and $d(I_j, I_k) \geq \frac{1}{2}|I_k|$

For $x \in F = \bigcup_{\nu} I_{\nu}$ decompose the double integral as the sum

$$\sum_{i,j} \int_{I_i} \int_{I_j} (1-r)^2 \int_{I_i} \frac{f(v)}{(1-r)^2 + c(v-x)^2} dv \int_{I_j} \frac{f(u)}{(1-r)^2 + c(u-v)^2} du$$

i) adjacent I_j : there are at most 2 of those intervals,

$$(1-r)^2 \int_{I_i} \frac{f(v)}{(1-r)^2 + c(v-x)^2} dv \int_{I_j} \frac{f(u)}{(1-r)^2 + c(u-v)^2} du \leq \int_{I_i} \frac{f(v)}{(v-x)^2} dv \times C\lambda|I_i| \leq C\lambda\mathcal{F}(x).$$

ii) non adjacent I_j :

$$(1-r)^2 \int_{I_i} \frac{f(v)}{(1-r)^2 + c(v-x)^2} dv \times \sum_j C \int_{I_j} \frac{\lambda}{(1-r)^2 + (v-x)^2} \leq C\lambda^2.$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, IL, 60607, USA.

E-mail address: [Calixto P. Calderón] cpc@uic.edu

DEPARTMENT OF MATHEMATICS, ILLINOIS STATE UNIVERSITY, NORMAL IL, 61761, USA.

E-mail address: [A. Susana Coré] bacore@ilstu.edu

DEPARTMENT OF MATHEMATICAL AND ACTUARIAL SCIENCES, ROOSEVELT UNIVERSITY CHICAGO, IL, 60605, USA.

E-mail address: [Wilfredo Urbina] wurbinaromero@roosevelt.edu